Centre Number | 1 | 2 | 5

CRANBROOK SCHOOL

Student Number | | | | | | | |

**2021**

HSC Examination

Assessment Task 4

# Extension 2 Mathematics

# Trial Examination

**General Instructions**

- Write using blue or black pen
- Diagrams drawn using pencil
- A Board-approved calculator may be used
- **Section 1**: Use the **Multiple Choice** Answer sheet for questions 1 to 10.
- **Section 2:** Please write each question in a new booklet.
- All relevant working should be shown for each question.

This paper must not be removed from the examination room

*Disclaimer*
*The content and format of this paper does not necessarily reflect the content and format of the HSC examination paper.*

# Section I – Multiple Choice

**10 Marks**

---

**1.** Which of the following points lies on the line described by the vector equation

$$r = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} ?$$

(A) $\begin{pmatrix} -3 \\ 9 \\ 1 \end{pmatrix}$

(B) $\begin{pmatrix} -3 \\ -8 \\ -3 \end{pmatrix}$

(C) $\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$

(D) $\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$

**2.** Which value of $z$ satisfy $z^2 = 7 + 24i$ ?

(A) $4 + 3i$

(B) $-4 + 3i$

(C) $-3 + 4i$

(D) $3 + 4i$

**3.** What is the angle between vectors $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + k$ and $\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} + 2k$, to the nearest degree?

(A) $77^\circ$

(B) $83^\circ$

(C) $84^\circ$

(D) $96^\circ$

**4.** $A(1,2,2)$, $B(3,-12,4)$, $C(1,2,0)$ and $D(3,-12,0)$ are four positional vectors.

What is the vector projection of $\overrightarrow{AB}$ onto $\overrightarrow{CD}$?

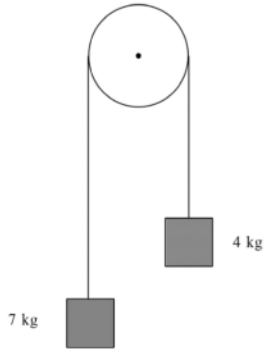
(A) $2\underset{\sim}{i} - 14\underset{\sim}{j} + 2\underset{\sim}{k}$

(B) $2\underset{\sim}{i} - 14\underset{\sim}{j} + 4\underset{\sim}{k}$

(C) $2\underset{\sim}{i} - 14\underset{\sim}{j}$

(D) $-2\underset{\sim}{i} + 14\underset{\sim}{j}$

**5.** A light inextensible string passes over a smooth pulley. Attached to each end of the strings are masses of 4 kg and 7 kg, as shown.




The acceleration of the larger mass downwards is

(A) $\dfrac{3g}{11}$

(B) $\dfrac{11g}{3}$

(C) $\dfrac{7g}{11}$

(D) $3g$

6. A particle is moving in simple harmonic motion with a displacement of $x$ metres. Its acceleration, $\ddot{x}$, is given by $\ddot{x} = -4x + 3$.

What are the centre and period of motion?

(A) Centre of motion $= 3$, period $= \dfrac{\pi}{2}$

(B) Centre of motion $= -3$, period $= \pi$

(C) Centre of motion $= \dfrac{3}{4}$, period $= \pi$

(D) Centre of motion $= \dfrac{3}{4}$, period $= \dfrac{\pi}{2}$

7. It is given that $z = 2 + i$ is a root of $z^3 + az^2 - bz + 5 = 0$, where $a$ and $b$ are real numbers.

What is the value of $a$?

(A) $-5$

(B) $-3$

(C) $3$

(D) $5$

**8.** Which integral has the smallest value?

(A) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x \, dx$

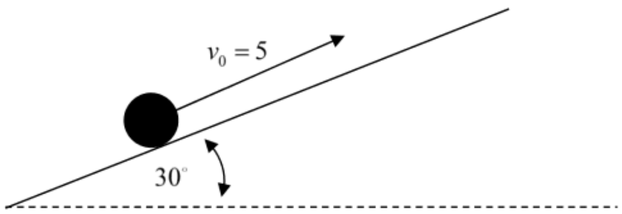
(B) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$

(C) $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \cos x \, dx$

(D) $\displaystyle\int_0^{\frac{\pi}{4}} \sin x \tan x \, dx$

**9.** A ball of unit mass is rolled up a frictionless ramp that is inclined at $30^\circ$ to the horizontal. It has an initial velocity of $5\ ms^{-1}$.

Assuming $g = 10\ ms^{-2}$, what is the net acceleration on the ball?




(A) $5\sqrt{3}\ ms^{-2}$ directed down the ramp.

(B) $5\sqrt{3}\ ms^{-2}$ directed up the ramp.

(C) $5\ ms^{-2}$ directed down the ramp.

(D) $5\ ms^{-2}$ directed up the ramp.

**10.** What value of $a$ will minimise the integral $\int_0^1 \left(x^2 - a\right)^2 \, dx$?

(A) $a = \dfrac{1}{2}$

(B) $a = \dfrac{1}{\sqrt{2}}$

(C) $a = \dfrac{4}{45}$

(D) $a = \dfrac{1}{3}$

**END OF MULTIPLE CHOICE**

# Section II – Extended response

**90 Marks**

---

**Question 11 – Please start a new booklet.** **15 Marks**

(a) Let $z = 4 - 3i$ and $w = 2 + 5i$, evaluate

(i) Show that $\dfrac{w}{\overline{z}} = \dfrac{23+14i}{25}$. **2**

(ii) Show that $\left(w+\overline{z}\right)\left(\overline{w}+z\right) = 100$. **2**

(b) Find the square roots of $15 - 8i$. Show all working. **3**

(c) Use the substitution $t = \tan\dfrac{\theta}{2}$ to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta+\cos\theta}$. **4**

(d) The acceleration, $a$, of a particle moving in a straight line is given by $a = x - 4$ where $x$ is the displacement in metres. The particle is initially at the origin and travelling with velocity of $2\ ms^{-1}$.

(i) Show that the velocity of the particle is described by $v^2 = x^2 - 8x + 4$. **2**

(ii) Find the acceleration of the particle when it comes to rest. **2**

**End of Question 11**

**Question 12 - Please start a new booklet.** **15 Marks**

(a) Use integration by parts to find $\int x3^x \, dx$ **3**

(b) By writing $\dfrac{8-2x}{(1+x)(4+x^2)}$ in the form $\dfrac{a}{1+x}+\dfrac{bx+c}{4+x^2}$, evaluate **4**

$$\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\,dx$$

.

(c)

(i) On the same Argand diagram, draw a neat sketch of $|z-4-4i|=2$ and

$\arg(z)=\dfrac{\pi}{4}$. **2**

(ii) Hence write down all the values of $z$ which satisfy $|z-4-4i|=2$ and $\arg(z)=\dfrac{\pi}{4}$

simultaneously. **2**

(d) Find the scalar projection of the vector $\underset{\sim}{u} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ onto the vector $4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$. **2**

(e) Given $\underset{\sim}{a} = \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$, and $\underset{\sim}{a} - \underset{\sim}{b} + 2\underset{\sim}{c} = 0$, find $\underset{\sim}{c}$. **2**

**End of Question 12**

**Question 13 – Please start a new booklet.** **15 Marks**

(a) Let $I_1 = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$ and $I_2 = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx$.

(i) Using the substitution $u = \pi - x$, show that $I_1 = I_2$ **2**

(ii) Hence, or otherwise, evaluate $I_1$. **2**

(b) A mass of 1 kg moves along a straight line with velocity $v$ $ms^{-1}$. It encounters a resistance of $v + v^3$. The particle has initial velocity $U$, where $U > 0$ and starts from the origin.

(i) Show that the equation of motion is $\ddot{x} = -v(1 + v^2)$. **1**

(ii) Show that $x = \tan^{-1}\left(\frac{U - v}{1 + Uv}\right)$. **2**

(iii) Show that $v^2 = \frac{U^2}{(1 + U^2)\, e^{2t} - U^2}$. **3**

(c) At time $t$ the particle has velocity $v$ and displacement $x$. A particle is travelling in a straight line. Its displacement, $x$ cm, from $O$ at a given time, $t$ seconds after the start of the motion, is given by $x = 3 + \sin^2 t$.

(i) Prove that the particle in undergoing simple harmonic motion. **2**

(ii) Find the period of the motion. **1**

(iii) Find the total distance travelled by the particle in the first $\pi$ seconds. **2**

**End of Question 13**

**Question 14 – Please start a new booklet.** **15 Marks**

(a) The scalar product of $\underset{\sim}{i} - 2\lambda \underset{\sim}{j} - \underset{\sim}{k}$ and the sum of $\underset{\sim}{i} - \lambda \underset{\sim}{k}$ and $\lambda \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, is 6.

Find $\lambda$. **2**

(b) Relative to the origin *O*, the points *A, B, C* and *D* have position vectors given respectively by $-4\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, $4\mathbf{i} + \lambda\mathbf{j} + 6\mathbf{k}$, $4\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $2\mathbf{j} - 6\mathbf{k}$.

(i) Given that the line $\overrightarrow{AC}$ is perpendicular to the line $\overrightarrow{BD}$, determine the value of $\lambda$. **2**

(ii) Hence find the position vector of *F*, the point of intersection of the lines $\overrightarrow{AC}$ and $\overrightarrow{BD}$. **3**

(c) Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = 3\underset{\sim}{a} + 2\underset{\sim}{b}$.

(i) Prove that if $\overrightarrow{OD} = \frac{1}{5}\overrightarrow{OC}$, the *D* lies on *AB*. **3**

(ii) Is the point *D* closer to point *A* or point *B*? Justify your answer. **1**

(d)

(i) Given $z = \cos\theta + i\sin\theta$, prove that $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$. **1**

(ii) Hence, by considering the expansion of $\left(z - \dfrac{1}{z}\right)^5$, show that **3**

$$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$$

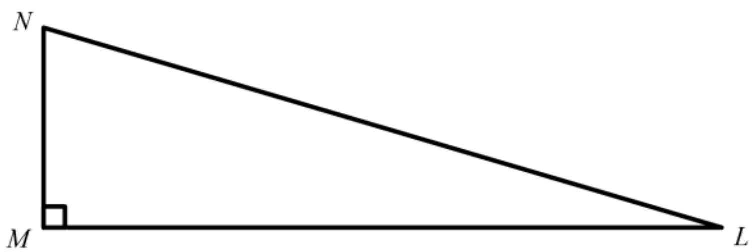
**End of Question 14**

**Question 15 – Please start a new booklet.** **15 Marks**

(a)

(i) Prove that for non-zero vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $(\underset{\sim}{a}+\underset{\sim}{b}).(\underset{\sim}{a}+\underset{\sim}{b})=|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2$ if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular. **2**

(ii) In $\Delta LMN$, let $\overrightarrow{LM}=\underset{\sim}{a}$ and $\overrightarrow{MN}=\underset{\sim}{b}$.




By finding an expression for the side $LN$ in terms of vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, or otherwise, prove that $|LN|^2=|LM|^2+|MN|^2$. **2**

(b) Find $\int x^2\sqrt{1-x^2}\ dx$ **3**

(c) A particle of unit mass is moving vertically downward in a medium which exerts a resistance force proportional to the square of the speed, $v$, of the particle. It is released from rest at $O$ and its terminal velocity is $U$.

(i) Show that the distance it has fallen below $O$ is given by **2**

$$x = \frac{1}{2k} \ln \left| \frac{g}{g - kv^2} \right| .$$

(ii) Prove that the time taken, $T$, for the particle to fall from $O$ to when its velocity is half its terminal velocity, $U$, is given by **3**

$$T = \frac{U}{2g} \ln 3 .$$

(d) Using integration by parts, calculate $\int (1 + 2x^2)\, e^{x^2}\, dx$ . You may wish to consider this integral as the sum of two integrals. **3**

**End of Question 15**

**Question 16 – Please start a new booklet.** **15 Marks**

(a)

(i) Find all the roots of $z^7 - 1 = 0$ in exponential form. **2**

(ii) Using $z^7 - 1 = (z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1)$, or otherwise,

prove that $\dfrac{2\pi}{7}$, $\dfrac{4\pi}{7}$ and $\dfrac{6\pi}{7}$ are solutions to **3**

$$2\cos 3\theta + 2\cos 2\theta + 2\cos\theta + 1 = 0$$

(iii) Hence, or otherwise, prove that **2**

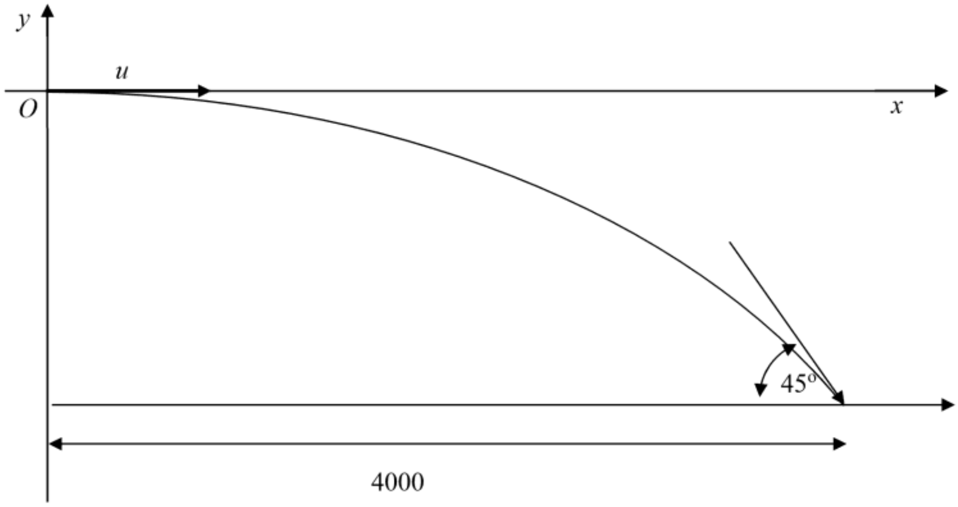
$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}.$$

(b) Let $I_{2n+1} = \displaystyle\int_0^1 x^{2n+1} e^{x^2}\, dx$ where $n$ is a positive integer.

(i) Prove that $I_{2n+1} = \dfrac{e}{2} - nI_{2n-1}$. **3**

(ii) Hence or otherwise, prove that $2\displaystyle\int_0^1 x^{2n-1}\left(1+x^2\right)e^{x^2}\, dx \le e$ for $n \ge 1$. **2**

(c) An aircraft flying horizontally at $u\ ms^{-1}$ delivers an emergency medical supply package that hits the ground 4000 metres away, measured horizontally. The package experiences an air resistance of $0.1v$ where $v$ is the velocity at time $t$ and $g$ is the acceleration due to gravity. The package hits the ground at an angle of $45^\circ$ to the horizontal.




You can assume that after $t$ seconds after release the position vector is given by

$$r(t)=\begin{pmatrix} 10u\left(1-e^{-0.1t}\right) \\ 100g\left(1-e^{-0.1t}\right)-10gt \end{pmatrix}. \qquad \text{(Do not prove this result)}$$

(i) Show that the velocity vector $\underset{\sim}{v}(t)$ of the particle is given by **1**

$$\underset{\sim}{v}(t)=\begin{pmatrix} ue^{-0.1t} \\ -10g\left(1-e^{-0.1t}\right) \end{pmatrix}.$$

(ii) Find the time when the package hits the ground and the speed on impact, where

$g = 10\, ms^{-2}$ . **2**

**End of Exam**

Ext 2 2021 Trial Solutions - Q13

Monday, 16 August 2021 12:38 pm

(a) $I_1 = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx \qquad I_2 = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}$

let $u = \pi - x \qquad x = \pi, u = 0$

$\frac{du}{dx} = -1 \qquad x = 0, u = \pi$

$$I_1 = \int_{\pi}^{0} \frac{(\pi - u)\sin(\pi - u)}{1 + [\cos(\pi - u)]^2} - du$$ (1)

flips limits.

$\sin(\pi - \theta) = \sin\theta$, $\cos(\pi - \theta) = -\cos\theta$ (This is an angle in the second quadrant)

$$= \int_0^{\pi} \frac{(\pi - u)\sin u}{1 + (-\cos u)^2}\,du$$

$$= \int_0^{\pi} \frac{(\pi - u)\sin u}{1 + \cos^2 u}\,du$$

As this is a definite integral, the variable of integration can be changed.

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\,dx \quad \text{As required}$$ (1)

(ii) $I_1 = I_2$

$$I_2 = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x} - \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$\therefore I_1 = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx - I_1$$

$$2I_1 = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx$$ (1)

let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\therefore -du = \sin x \, dx$

$x = \pi, \quad u = -1$

$x = 0 \quad u = 1$

$$= -\frac{\pi}{2} \int \frac{1}{1+u^2} \, du$$

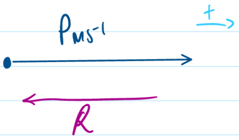
$$= -\frac{\pi}{2} \left[ \tan^{-1}(u) \right]_1^{-1} + C$$

$$= -\frac{\pi}{2} \left[ -\frac{\pi}{4} - \frac{\pi}{4} \right]$$

$$= \frac{\pi^2}{4}$$

1

(b) (i)




let $u = P$

$R = v + v^3$

1

$\sum = -R$

$m\ddot{x} = -(v + v^3) \qquad m = 1$

$\therefore \ddot{x} = -v(1+v^2)$

(ii) $v \cdot \frac{dv}{dx} = \ddot{x}$

$\therefore \cancel{v} \cdot \frac{dv}{dx} = -\cancel{v}(1+v^2)$

$\frac{-dv}{1+v^2} = dx$

$$\int_P^v \frac{-dv}{1+v^2} = \int_0^x dx$$

$\therefore x = -\left[ \tan^{-1}(v) \right]_P^v$ 1

$= \underset{B}{\underbrace{-\tan^{-1}(v)}} + \underset{A}{\underbrace{\tan^{-1}(P)}}$

$= \tan^{-1}\left( \frac{P - v}{1 + Pv} \right)$ As required.

$\tan\alpha = A \qquad \tan\beta = B$

$\therefore \alpha + \beta = \tan^{-1}A - \tan^{-1}B$

$\tan(\alpha+\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$= \frac{A - B}{1 + AB}$

$\therefore \alpha + \beta = \tan^{-1}\left( \frac{A-B}{1+AB} \right)$

1

(iii) $\ddot{x} = \frac{dv}{dt}$

$$\therefore \quad \frac{dv}{dt} = -v(1+v^2)$$

$$\therefore \quad dt = \frac{-dv}{v(1+v^2)}$$

$$\int_0^t dt = -\int_p^v \frac{1}{v(1+v^2)}\,dv \quad 1$$

let $\frac{A}{v} + \frac{Bv+C}{1+v^2} = \frac{1}{v(1+v^2)}$

$\therefore \quad 1 = A(1+v^2) + v(Bv+C)$

When $v=0 \quad 1 = A$

$v=1 \quad 1 = 2A + B + C \qquad \therefore B+C=-1$

$v=-1 \quad 1 = 2A + B - C \qquad B-C=-1$

$\therefore \quad C=0$
$B=-1$

$$t = -\int_p^v \frac{1}{v} - \frac{v}{1+v^2}\,dv \quad 1$$

$$= -\left[\ln v - \frac{1}{2}\ln|1+v^2|\right]_p^v$$

$$= \frac{1}{2}\left[\ln|1+v^2| - \ln v^2\right]_p^v$$

$$= \frac{1}{2}\left[\ln\left|\frac{1+v^2}{v^2}\right|\right]_p^v$$

$$= \frac{1}{2}\left[\ln\left|\frac{1+v^2}{v^2}\right| - \ln\left|\frac{1+p^2}{p^2}\right|\right]$$

$$t = \frac{1}{2}\ln\left|\frac{p^2(1+v^2)}{v^2(1+p^2)}\right|$$

$$\therefore \quad \frac{p^2(1+v^2)}{v^2(1+p^2)} = e^{2t}$$

$$p^2(1+v^2) = v^2(1+p^2)e^{2t}$$

$$p^2 + p^2v^2 = v^2(1+p^2)e^{2t}$$

$$p^2 = v^2(1+p^2)e^{2t} - p^2v^2$$

$$= v^2\left([1+p^2]e^{2t} - p^2\right)$$

1

$$v^2 = \frac{p^2}{(1+p^2)e^{2t} - p^2} \qquad \text{As required.}$$

(c) (i) $x = 3+\sin^2 t \qquad \therefore \sin^2 t = x-3$

$$\frac{dx}{dt} = 2\cos t \sin t$$

$$\frac{d^2x}{dt^2} = 2\cos t \times \cos t + \sin t \times -2\sin t$$

$$= 2\cos^2 t - 2\sin^2 t$$

$\ddot{x}$

$= 2\cos^2 t - 2\sin^2 t$

$= 2(1-\sin^2 t) - 2\sin^2 t$

$= 2 - 4\sin^2 t$

$= 2 - 4(x-3)$

$= 2 - 4x + 12$

$= 14 - 4x$

$= 4\left[\frac{7}{2} - x\right]$

1

This SHM as the acceleration is proportional to, but in the opposite direction of, the displacement from $\frac{7}{2}$. 1

(ii) $n = 2$ $\therefore T = \frac{2\pi}{2}$

$= \pi$

(iii) $\pi$ seconds is one period $\therefore$ It will travel four times the amplitude.

If $x = 3 + \sin^2 t$ then max value for $x = 4$ & min value is 3

1 Justifying the answer.

$\therefore$ Amplitude is $\frac{1}{2}$ cm

$\therefore$ Travels 2 cm 1

(a) sum of $\underset{\sim}{i} - \lambda \underset{\sim}{k}$ & $\lambda \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ = $(\lambda+1)\underset{\sim}{i} + 2\underset{\sim}{j} + (-1-\lambda)\underset{\sim}{k}$

$$\begin{pmatrix} 1 \\ -2\lambda \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \lambda+1 \\ 2 \\ -1-\lambda \end{pmatrix} = \lambda+1 \ -4\lambda + 1 + \lambda$$

1

$6 = -2\lambda + 2$
$-2\lambda = 4$
$\lambda = -2$ 1

(b) (i) If $\overrightarrow{AC}$ perpendicular to $\overrightarrow{BD}$ then $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

$$\overrightarrow{AC} = \begin{pmatrix} 4-(-4) \\ -1 \ -3 \\ -1 \ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ -4 \end{pmatrix}$$

$$\overrightarrow{BD} = \begin{pmatrix} 0 - 4 \\ 2 - \lambda \\ -6 - 6 \end{pmatrix} = \begin{pmatrix} -4 \\ 2-\lambda \\ -12 \end{pmatrix}$$ 1

$\overrightarrow{AC} \cdot \overrightarrow{BD} = 8\times-4 + (-4)(2-\lambda) + (-4)(-12)$
$= -32 - 8 + 4\lambda + 48$
$0 = 8 + 4\lambda$
$4\lambda = -8$
$\lambda = -2$ 1

(ii) The line $\overrightarrow{AC} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 8 \\ -4 \\ -4 \end{pmatrix}$

The line $\overrightarrow{BD} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} -4 \\ 4 \\ -12 \end{pmatrix}$ 1

Solve simultaneously $x$: $-4 + 8\lambda_1 = 4 - 4\lambda_2$
$-1 + 2\lambda_1 = 1 - \lambda_2$
$2\lambda_1 + \lambda_2 = 2$

Solve

$y$: $3 - 4\lambda_1 = -2 + 4\lambda_2$
$5 = 4\lambda_1 + 4\lambda_2$

$2\lambda_1 + \lambda_2 = 2$ $\quad 4\lambda_1 + 4\lambda_2 = 5$ ... (2)
$8\lambda_1 + 4\lambda_2 = 8$ ... (1)
(1) − (2)
$4\lambda_1 = 3$
$\lambda_1 = \frac{3}{4}$ $\quad \therefore \lambda_2 = \frac{1}{2}$ 1

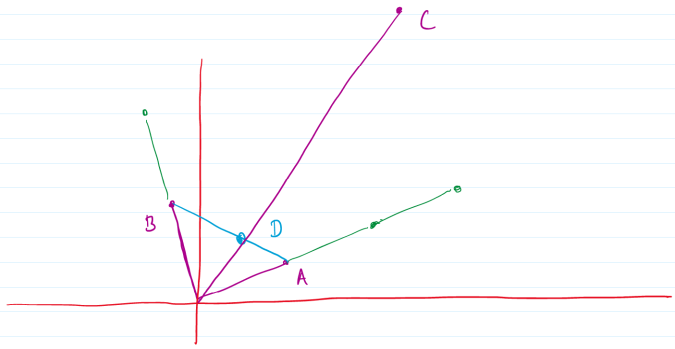
Check with $z$: $3 - 4\lambda_1 = 6 - 12\lambda_2$
$3 - 3 = 6 - 6$
$0 = 0$

$\therefore \lambda_1 = \frac{3}{4}$ & $\lambda_2 = \frac{1}{2}$ at the point of intersection.

$\therefore \underset{\sim}{F} = \begin{pmatrix} -4+8\lambda_1 \\ 3-4\lambda_1 \\ 3-4\lambda_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ $\Leftarrow$ $\underset{\sim}{f} = 2\underset{\sim}{i}$ 1

(c) (i) $\overrightarrow{OA} = \underset{\sim}{a}$ $\quad \overrightarrow{OB} = \underset{\sim}{b}$ $\quad \overrightarrow{OC} = 3\underset{\sim}{a} + 2\underset{\sim}{b}$
C

(c) (i) $\overrightarrow{OA} = \underset{\sim}{a}$ $\quad \overrightarrow{OB} = \underset{\sim}{b}$ $\quad \overrightarrow{OC} = 3\underset{\sim}{a} + 2\underset{\sim}{b}$




$\overrightarrow{OD} = \frac{1}{5}\left(3\underset{\sim}{a} + 2\underset{\sim}{b}\right)$

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$ (1)

$\overrightarrow{AD} = \frac{1}{5}\left(3\underset{\sim}{a} + 2\underset{\sim}{b}\right) - \underset{\sim}{a}$

$= -\frac{2}{5}\underset{\sim}{a} + \frac{2}{5}\underset{\sim}{b}$

$= \frac{2}{5}\left(\underset{\sim}{b} - \underset{\sim}{a}\right)$ (1)

$= \frac{2}{5}\overrightarrow{AB}$ $\therefore$ D lies on $\overrightarrow{AB}$ As $\overrightarrow{AD}$ is parallel to $\overrightarrow{AB}$ but shorter (multiplied by a constant that is $0 < k < 1$) (1)

(ii) $\overrightarrow{AD} = \frac{2}{5}\overrightarrow{AB}$ which means $\overrightarrow{DB} = \frac{3}{5}\overrightarrow{AB}$

$\therefore$ D is closer to A as $|\overrightarrow{AD}| < |\overrightarrow{BD}|$ (1)

(d) (i) $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$ (By DMT)

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$ But, $\cos\theta$ is an even function & $\sin\theta$ is an odd function

$z^{-n} = \cos(n\theta) - i\sin(n\theta)$

$\therefore z^n + z^{-n} = \cancel{\cos(n\theta)} + i\sin(n\theta) - \left[\cancel{\cos(n\theta)} - i\sin(n\theta)\right]$

$= 2i\sin(n\theta)$ As required. (1)

(ii) $\left(z - \frac{1}{z}\right)^5 = z^5 - 5\times z^4 \times \frac{1}{z} + 10\times z^3 \times \frac{1}{z^2} - 10 \times z^2 \times \frac{1}{z^3} + 5\times z \times \frac{1}{z^4} - \frac{1}{z^5}$

$= \left(z^5 - \frac{1}{z^5}\right) - 5\left[z^3 - \frac{1}{z^3}\right] + 10\left[z - \frac{1}{z}\right]$ (1)

$= 2i\sin 5\theta - 5\left[2i\sin 3\theta\right] + 10\left[2i\sin\theta\right]$

$$= 2i\sin 5\theta - 5\left[2i\sin 3\theta\right] + 10\left[2i\sin\theta\right]$$

$$= 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \qquad 1$$

$$\left(z - \frac{1}{z}\right)^5 = \left(2i\sin\theta\right)^5$$

$$= 32i\sin^5\theta$$

$$\therefore \quad 32\cancel{i}\sin^5\theta = 2\cancel{i}\sin 5\theta - 10\cancel{i}\sin 3\theta + 20\cancel{i}\sin\theta \qquad 1$$

$$\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta \quad \text{As required.}$$

(a) (i) $(\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b}) = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$

LHS $= \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{b}$ (1)

if $\underset{\sim}{a} \perp \underset{\sim}{b}$ then $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ (1)

$= \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{b}$

$\underset{\sim}{a}\cdot\underset{\sim}{a} = |\underset{\sim}{a}|^2$

$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$ As required.

(ii)


$\overrightarrow{LM} = \underset{\sim}{a}$ $\quad \overrightarrow{MN} = \underset{\sim}{b}$ $\quad$ $\underset{\sim}{a}$ & $\underset{\sim}{b}$ are perpendicular.

$\therefore \overrightarrow{LN} = \underset{\sim}{a} + \underset{\sim}{b}$

$|\overrightarrow{LN}|^2 = \overrightarrow{LN}\cdot\overrightarrow{LN}$ (1)

$= (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b})$

$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$ (see part (i)) (1)

$= |\overrightarrow{LM}|^2 + |\overrightarrow{MN}|^2$ As required.

(b) $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

let $x = \sin\theta$

$\dfrac{dx}{d\theta} = \cos\theta$

$\therefore I = \displaystyle\int \sin^2\theta\sqrt{1-\sin^2\theta}\,d\theta\cdot\cos\theta$ (1)

$= \displaystyle\int \sin^2\theta\cos^2\theta\,d\theta$

$= \displaystyle\int (\sin\theta\cos\theta)^2\,d\theta$

$= \displaystyle\int \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$

$= \dfrac{1}{4}\displaystyle\int \sin^2 2\theta\,d\theta$ $\qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

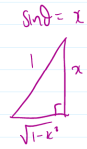
$= \dfrac{1}{8}\displaystyle\int 1 - \cos 4\theta\,d\theta$ $\qquad \therefore \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ (1)

$= \dfrac{1}{8}\left[\theta - \dfrac{\sin 4\theta}{4}\right] + C$

$\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$= 2\left[2\sin\theta\cos\theta\right]\left[\cos^2\theta - \sin^2\theta\right]$

if $\sin\theta = x$ $\quad \cos\theta = \sqrt{1-x^2}$




$= \dfrac{1}{8}\left[\sin^{-1}(x) - 4\left(x\sqrt{1-x^2}\right)\left(1-x^2-x^2\right)\right]$

$= \dfrac{1}{8}\left[\sin^{-1}(x) - 4x\sqrt{1-x^2}\left(1-2x^2\right)\right] + C$ (1) In terms of $x$

(c) (i)


$R = kv^2$

$$\sum F = mg - R$$

$$\therefore m\ddot{x} = mg - R \qquad \text{But } m=1$$

$$\therefore \ddot{x} = g - R$$

$$v\cdot\frac{dv}{dx} = g - kv^2$$

$$\therefore dx = \frac{v}{g-kv^2}\,dv \qquad (1)$$

$$\int_0^x dx = \int_0^v \frac{v}{g-kv^2}\,dv$$

$$x = -\frac{1}{2k}\int_0^v \frac{2kv}{g-kv^2}\,dv$$

$$= -\frac{1}{2k}\Big[\ln|g-kv^2|\Big]_0^v$$

$$= -\frac{1}{2k}\Big[\ln|g-kv^2| - \ln|g|\Big]$$

$$= \frac{1}{2k}\ln\left|\frac{g}{g-kv^2}\right| \qquad \text{As required.}$$

(1)

(ii)

$$\frac{dv}{dt} = g - kv^2$$

$$\therefore dt = \frac{1}{g-kv^2}\,dv$$

let $\frac{1}{(\sqrt{g}-\sqrt{k}v)(\sqrt{g}+\sqrt{k}v)} = \frac{A}{\sqrt{g}+\sqrt{k}v} + \frac{B}{(\sqrt{g}-\sqrt{k}v)}$

$$1 = A(\sqrt{g}-\sqrt{k}v) + B(\sqrt{g}+\sqrt{k}v)$$

When $v=0$ $\quad 1 = \sqrt{g}(A+B) \quad \cdots (1)$

When $v=1$ $\quad 1 = \sqrt{g}(A+B) + \sqrt{k}(B-A) \quad \cdots (2)$

$(2)-(1)$

$$\sqrt{k}(B-A) = 0$$

$$\therefore B = A$$

$$\therefore 2\sqrt{g}A = 1$$

$$A = \frac{1}{2\sqrt{g}}$$

$$\therefore \frac{1}{g-kv^2} = \frac{1}{2\sqrt{g}}\left(\frac{1}{\sqrt{g}+\sqrt{k}v} + \frac{1}{\sqrt{g}-\sqrt{k}v}\right)$$

$$\int_0^t dt = \int_0^v \frac{1}{2\sqrt{g}}\left(\frac{1}{\sqrt{g}+\sqrt{k}v} + \frac{1}{\sqrt{g}-\sqrt{k}v}\right)dv$$

$$t = \frac{1}{2\sqrt{kg}}\left[\ln\left|\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right|\right]_0^v$$

$$= \frac{1}{2\sqrt{kg}}\ln\left|\frac{\sqrt{g}+\sqrt{k}v}{\sqrt{g}-\sqrt{k}v}\right| \qquad (1)$$

Find $t$ when $v = \frac{U}{2}$ (let $U = p$)

$$\therefore t = \frac{1}{2\sqrt{kg}}\ln\left|\frac{\sqrt{g}+\sqrt{k}\cdot\frac{P}{2}}{\sqrt{g}-\sqrt{k}\cdot\frac{P}{2}}\right| \qquad (1)$$

However, As $P$ is terminal velocity

$kP^2 = g$ (As terminal velocity occurs when $\sum F = 0$)

$$P^2 = \frac{g}{k}$$

$\because p = \sqrt{\frac{g}{k}}$

$$t = \frac{1}{2\sqrt{kg}} \ln\left[\frac{\sqrt{g} + \sqrt{k}\cdot\frac{p}{2}}{\sqrt{g} - \sqrt{k}\cdot\frac{p}{2}}\right] \begin{matrix} \div\sqrt{k} \\ \div\sqrt{k} \end{matrix}$$

$$= \frac{1}{2\sqrt{kg}} \ln\left[\frac{\sqrt{\frac{g}{k}} + \frac{p}{2}}{\sqrt{\frac{g}{k}} - \frac{p}{2}}\right]$$

$$= \frac{1}{2\sqrt{kg}} \ln\left[\frac{p + \frac{p}{2}}{p - \frac{p}{2}}\right]$$

$$= \frac{1}{2\sqrt{kg}} \ln\left|\frac{\frac{3\cancel{p}}{\cancel{2}}}{\frac{\cancel{p}}{\cancel{2}}}\right|$$

$$= \frac{1}{2\sqrt{kg}} \ln 3$$

$$= \frac{1}{2} \times \sqrt{\frac{g}{k}} \times \frac{1}{g} \ln 3$$

$$= \frac{p}{2g} \ln 3 \quad \text{As required.}$$

1

(d) $\int (1+2x^2)e^{x^2}\,dx = \int e^{x^2}\,dx + \int 2x^2 e^{x^2}\,dx$

$I = \int e^{x^2}\,dx \qquad J = \int 2x^2 e^{x^2}\,dx$ 1

$u = x \qquad v = e^{x^2}$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2xe^{x^2}$

$J = xe^{x^2} - \int e^{x^2}\,dx$ 1

$= xe^{x^2} - I$

1

$\therefore J + I = xe^{x^2} + C$

(a) (i) $z^7 = 1$ let $z = e^{i\theta}$ As $|z|=1$

$e^{i7\theta} = e^0, e^{\pm 2\pi i}, e^{\pm 4\pi i}, e^{\pm 6\pi i}$

$\therefore e^{i\theta} = e^0, e^{\pm \frac{2\pi}{7}i}, e^{\pm\frac{4\pi}{7}i}, e^{\pm\frac{6\pi}{7}i}$ ① Answers ① Exp. form

(ii) let $z = \cos\theta + i\sin\theta$ $(|z|=1)$

$\left.\begin{aligned} z^n &= \cos n\theta + i\sin n\theta \\ z^{-n} &= \cos(-n\theta) + i\sin(-n\theta) \end{aligned}\right\}$ By DMT

But, cos is an even function & sin is an odd function

$\therefore z^{-n} = \cos n\theta - i\sin n\theta$

$z^n + z^{-n} = \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}$

$= 2\cos n\theta$ ①

NB: Can be done by conjugates.

$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ has roots $e^{\pm\frac{2\pi}{7}i}, e^{\pm\frac{4\pi}{7}i}, e^{\pm\frac{6\pi}{7}i}$

$\div z^3$ $(z \neq 0)$

$z^3 + z^2 + z + 1 + z^{-1} + z^{-2} + z^{-3} = 0$ ①

LHS $= z^3 + z^{-3} + z^2 + z^{-2} + z + z^{-1} + 1$

Using ①

$= (2\cos 3\theta) + (2\cos 2\theta) + (2\cos\theta) + 1 = 0$ As required ①

(iii) $2\cos 3\theta + 2\cos 2\theta + 2\cos\theta = -1$

$\cos 3\theta + \cos 2\theta + \cos\theta = -\frac{1}{2}$

Solutions are $\theta = \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{6}$

let $\theta = \frac{2\pi}{7}$

$\cos 3\left(\frac{2\pi}{7}\right) + \cos 2\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right) = -\frac{1}{2}$ ① substitution

(As $\theta = \frac{2\pi}{7}$ is a solution) ① Rearrange

$\cos\frac{6\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{2\pi}{7} = -\frac{1}{2}$ As required.

(b) (i) $I_{2n+1} = \int^1 x^{2n+1} e^{x^2}\,dx$

(b) (i) $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2} \, dx$

$= \int_0^1 x^{2n} \cdot x e^{x^2} \, dx$

$u = x^{2n}$ $\qquad v = \frac{e^{x^2}}{2}$

$\frac{du}{dx} = 2n x^{2n-1}$ $\qquad \frac{dv}{dx} = x e^{x^2}$

$\therefore I_{2n+1} = \left[ \frac{x^{2n} e^{x^2}}{2} \right]_0^1 - \int_0^1 \frac{\cancel{2}n \, x^{2n-1} e^{x^2}}{\cancel{2}} \, dx$

$= \left[ \frac{1 \times e}{2} - \frac{0 \times 1}{2} \right] - n \int x^{2n-1} e^{x^2} \, dx$

$= \frac{e}{2} - n I_{2n-1}$ As required.

(ii) $I_{2n+1} = \frac{e}{2} - n I_{2n-1}$

$\therefore I_{2n+1} + n I_{2n-1} = e/2$

As $n \geq 1$

$I_{2n+1} + I_{2n-1} \leq I_{2n+1} - n I_{2n-1}$
$\leq \frac{e}{2}$ ①

LHS $= \left[ \int_0^1 x^{2n+1} e^{x^2} \, dx + \int_0^1 x^{2n-1} e^{x^2} \, dx \right]$

$= \int_0^1 x^{2n+1} e^{x^2} + x^{2n-1} e^{x^2} \, dx$

$= \int_0^1 x^{2n-1} \left[ x^2 e^{x^2} + e^{x^2} \right] dx$

$= \int_0^1 x^{2n-1} e^{x^2} \left( 1 + x^2 \right) dx$ ①

$\therefore \int_0^1 x^{2n-1} e^{x^2} \left( 1 + x^2 \right) dx \leq \frac{e}{2}$

$$\therefore \int_0^1 x^{2n-1}e^{x^2}(1+x^2)\,dx \le \frac{e}{2}$$

$$2\int_0^1 x^{2n-1}e^{x^2}(1+x^2)\,dx \le e \quad \text{As required.}$$

(c) (i) $$\underset{\sim}{r}(t) = \begin{cases} 10u(1-e^{-0.1t}) \\ 100g(1-e^{-0.1t}) - 10gt \end{cases}$$

$$\frac{d}{dt}(\underset{\sim}{r}(t)) = \underset{\sim}{v}(t)$$

$$\frac{d}{dt}\, 10u(1-e^{-0.1t}) = 10u - 10ue^{-0.1t}$$

$$= 0 - \cancel{10}u \times -\cancel{0}.1\, e^{0.1t}$$

$$= ue^{0.1t}$$

$$\frac{d}{dt}\, 100g(1-e^{-0.1t}) - 10gt = \frac{d}{dt}\left(100g - 100ge^{-0.1t} - 10gt\right)$$

$$= 0 \cancel{-} 100g \times \cancel{-}0.1\, e^{-0.1t} - 10g$$

$$= 10ge^{-0.1t} - 10g$$

$$= -10g\left[1 - e^{-0.1t}\right]$$

1

(ii) If it hits the ground at 45° then
Horizontal velocity = Vertical velocity

$$ue^{-0.1t} = 100(1-e^{-0.1t})$$ (NB, changed to positive as it is travelling down.)

$$ue^{-0.1t} = 100 - 100e^{-0.1t}$$

$$\therefore (u+100)e^{-0.1t} = 100$$

$$e^{-0.1t} = \frac{100}{u+100} \quad \ldots (1)$$

When it hits the ground, it has travelled 4000 m horizontally. $\therefore x(t) = 4000$

$$4000 = 10u(1-e^{-0.1t}) \quad \ldots (2)$$

(1) $\Rightarrow$ (2)

$$400\cancel{0} = 1\cancel{0}u\left[1 - \frac{100}{u+100}\right]$$

$$= 10u\left[\frac{u+\cancel{100}-\cancel{100}}{u+100}\right]$$

$$400u + 40000 = u^2$$

$$u^2 - 400u - 40000 = 0$$

$$u = \frac{400 \pm \sqrt{(400)^2 + 4\times1\times40000}}{2}$$

$$= \frac{400 \pm 400\sqrt{2}}{2}$$

$$= 200 \pm 200\sqrt{2} \quad (\text{As } u>0)$$

$$u = 200 + 200\sqrt{2}$$

$$t = -10 \ln\left|\frac{100}{100+u}\right|$$

$$\doteqdot 17.6 \text{ seconds} \quad (1dp) \quad 1$$

let speed of impact $= v$

$$|v|^2 = |\dot{x}|^2 + |\dot{y}|^2 \qquad \text{But, } |\dot{x}| = |\dot{y}|$$

$$\therefore |\underset{\sim}{v}|^2 = 2|\dot{x}|^2$$

$$v = \sqrt{2}\left[(200+200\sqrt{2})e^{-0.1t}\right]$$

$$\doteqdot 117.16 \text{ ms}^{-1} \quad (2dp) \quad 1$$